Exercise 39

In the study of ecosystems, *predator-prey models* are often used to study the interaction between species. Consider populations of tundra wolves, given by $W(t)$, and caribou, given by $C(t)$, in northern Canada. The interaction has been modeled by the equations

$$\frac{dC}{dt} = aC - bCW \qquad \frac{dW}{dt} = -cW + dCW$$

(a) What values of dC/dt and dW/dt correspond to stable populations?

(b) How would the statement "The caribou go extinct" be represented mathematically?

(c) Suppose that $a = 0.05$, $b = 0.001$, $c = 0.05$, and $d = 0.0001$. Find all population pairs (C, W) that lead to stable populations. According to this model, is it possible for the two species to live in balance or will one or both species become extinct?

Solution

Part (a)

Since dC/dt and dW/dt represent the rates of caribou and wolf population growth, respectively, as t increases, stable populations are ones for which

$$\frac{dC}{dt} = 0 \quad \text{and} \quad \frac{dW}{dt} = 0.$$

Part (b)

The caribou go extinct when the caribou population reaches zero:

$$C(t) = 0.$$

Part (c)

Set $a = 0.05$, $b = 0.001$, $c = 0.05$, and $d = 0.0001$ in the model equations.

$$\frac{dC}{dt} = 0.05C - 0.001CW \qquad \frac{dW}{dt} = -0.05W + 0.0001CW$$

As stated in part (a), stable populations of caribou and wolves are ones with zero rates of change.

$$0 = 0.05C - 0.001CW \qquad 0 = -0.05W + 0.0001CW$$

$$0 = C(0.05 - 0.001W) \qquad 0 = (-0.05 + 0.0001C)W$$

These equations are satisfied if either $C = 0$ and $W = 0$, or

$$0.05 - 0.001W = 0 \quad \text{and} \quad -0.05 + 0.0001C = 0$$

$$W = 50 \quad \text{and} \quad C = 500.$$

Therefore, it is possible for the caribou and wolves to live in balance as long as there are 50 wolves and 500 caribou in the ecosystem.